2016

YEAR 12

HSC ASSESSMENT TASK 3

Mathematics Extension 2

Date: Tuesday 21st June Day 7 Period 0

Weighting: 10% of the HSC assessment: 5% Concepts, Skills and Techniques
5% Reasoning and Communication

General Instructions

- Reading time – 2 minutes
- Working time – 45 minutes
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided
- In questions 5-6 show relevant mathematical reasoning and /or calculations

Total marks – 33

Section 1: **4 marks**
- Attempt Questions 1-4
- Allow about 6 minutes for this section

Section 2: **29 marks**
- Attempt Questions 5-6
- Allow about 39 minutes for this section

Student Name………………….

Student Number……………….

Outcomes to be assessed:

E2 A student chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

E7 A student uses the techniques of slicing and cylindrical shells to determine volumes

E8 A student applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems

E9 A student communicates abstract ideas and relationships using appropriate notation and logical argument

Section I

4 marks
Attempt Questions 1 – 4
Answer Section I on the Multiple Choice Answer Sheet provided

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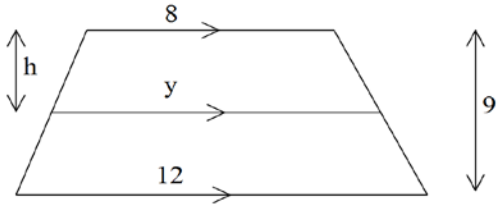
1 The diagram shows a trapezium with internal parallel line.



Which of the following is true?

(A) $y = \frac{3}{4}h + 8$

(B) $y = \frac{3}{4}h + 9$

(C) $4y = 9h + 72$

(D) $9y = 4h + 72$

2 Using an appropriate substitution which of the following is equivalent to

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x}{(1+\tan x)^2}\, dx?$$

(A) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{u}\, du$

(B) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{u^3}\, du$

(C) $\int_{0}^{2} \frac{1}{u^2}\, du$

(D) $\int_{0}^{2} \frac{u^2}{(1+u)^3}\, du$

3 Which is the correct answer to the following integral?

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^7 x \cos^4 x \, dx$$

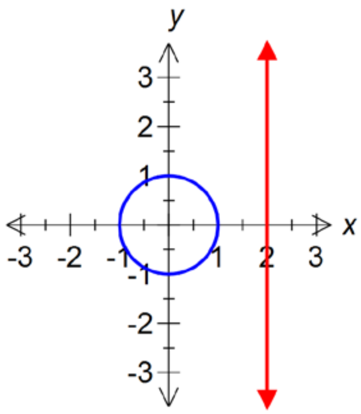
(A) $2 \times \int_0^{\frac{\pi}{4}} \sin^7 x \cos^4 x \, dx$

(B) 0

(C) $\dfrac{1024 - 533\sqrt{2}}{36\ 960}$

(D) $\dfrac{533\sqrt{2} - 1024}{36\ 960}$

4 The circle $x^2 + y^2 = 1$ is rotated about the line $x = 2$. What is the correct expression for the volume of this solid using the method of cylindrical shells?



(A) $V = 2\pi \int_{-1}^{1} (2 - x)\sqrt{1 - x^2} \, dx$

(B) $V = 4\pi \int_{-1}^{1} (2 - x)\sqrt{1 - x^2} \, dx$

(C) $V = 4\pi \int_{1}^{2} (2 - x)\sqrt{1 - x^2} \, dx$

(D) $V = 8\pi \int_{0}^{1} (2 - x)\sqrt{1 - x^2} \, dx$

Section II

29 marks
Attempt questions 5 – 6
Answer each question in the booklets provided.
In questions 5 - 6 show relevant mathematical reasoning and / or calculations.

Question 5 (15 marks) Start a new booklet

(a) By completing the square, or otherwise find $\int_{-1}^{5} \frac{dx}{\sqrt{32+4x-x^2}}$ **3**

(b) Find $\int x \tan^{-1} x \; dx$ **3**

(c) Find the volume of the solid of revolution generated when the area enclosed between the curve $y = 9 - x^2$ and the lines $y = 9$ and $x = 3$ is rotated about the $y = 9$. **3**

(d) Let $I_n = \int x^n e^x \; dx$

(i) Show that $I_n = x^n e^x - nI_{n-1}$ **1**

(ii) Hence evaluate $\int_1^2 x^2 e^x \; dx$ **2**

(e) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\sin x + 3\cos x}$ **3**

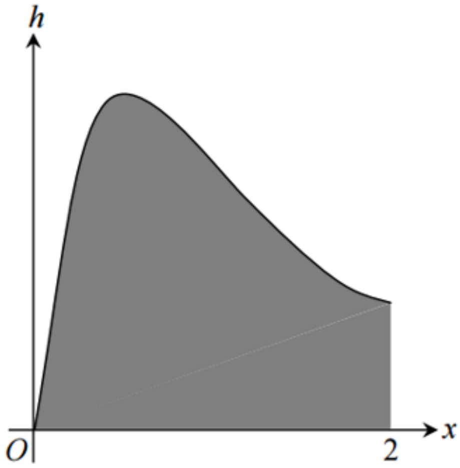
End of Question 5

Question 6 (14 marks) Start a new booklet

(a) Loreto as part of its Master Plan is planning to construct an artwork from 2 m wide panels. One side of each panel needs to be painted.

To determine the amount of paint needed, the area of each panel needs to be calculated.

The Visual Arts department makes a sketch of the panel.



The Mathematics department examines the sketches and decides the height of each panel can be modeled by the following function.

$$h(x)=\frac{10x}{(x^2+1)(3x+1)},\ 0\le x\le 2.$$

(i) Given that $\dfrac{10x}{(x^2+1)(3x+1)}$ can be written in the form $\dfrac{x+A}{x^2+1}+\dfrac{B}{3x+1}$ find the values of A and B. **2**

(ii) Hence, find the area of one of the panels correct to 2 decimal places. **3**

(b) (i) If $x\ge 0$, show that $\dfrac{x}{x^2+4}\le\dfrac{1}{4}$. **2**

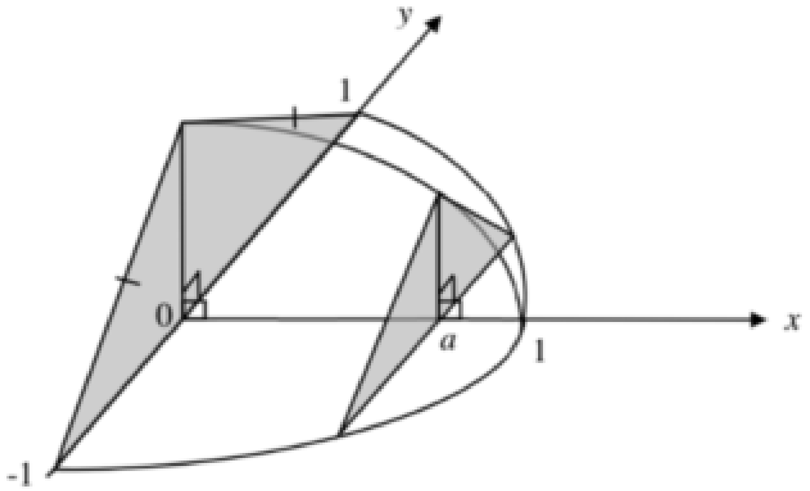
(ii) By integrating both sides of the inequality with respect to x between the limits of $x=0$ and $x=\alpha$, show that **2**

$$e^{\frac{1}{2}\alpha}\ge\frac{1}{4}\alpha^2+1 \text{ for } \alpha\ge 0.$$

Question 6 continues on the following page.

Question 6 continued

(c) The base of a solid is the semi-circular region of radius 1 unit in the x-y plane as illustrated below.



Each cross-section perpendicular to the x axis is an isosceles triangle with each of the two equal sides three quarters the length of the third side.

(i) Show that the area of the triangular cross-section at $x = a$ is **3**

$$\frac{\sqrt{5}}{2}\left(1-a^2\right).$$

(ii) Hence find the volume of the solid. **2**

End of Assessment Task

Mathematics Extension II

Multiple Choice Answer Sheet

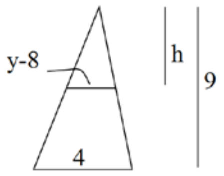
Student Number: ____________________________

1	A ⬭	B ⬭	C ⬭	D ⬭
2	A ⬭	B ⬭	C ⬭	D ⬭
3	A ⬭	B ⬭	C ⬭	D ⬭
4	A ⬭	B ⬭	C ⬭	D ⬭

2016 X2 Assessment Task 3 Solutions

<table>
<tr><td>

1.



$$\frac{y-8}{h}=\frac{4}{9}$$
$$9(y-8)=4h$$
$$9y-72=4h$$
$$9y=4h+72$$

</td><td>D</td></tr>
<tr><td>

2. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{sec^2 x}{(1+\tan x)^2}dx$

$Let\ \ u=1+\tan x$

$du=\sec^2 x dx$

$x=\frac{\pi}{4} \quad u=2$

$x=-\frac{\pi}{4} \quad u=0$

$\therefore \int_0^2 \frac{1}{u^2}du$

</td><td>C</td></tr>
<tr><td>

3. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^7 x \cos^4 x\, dx$

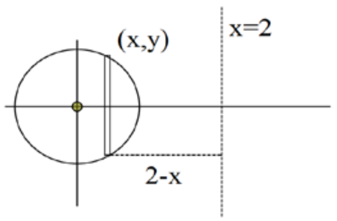
$f(x)=\sin^7 x$ odd fn

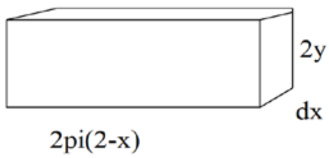
$\mathrm{g}(x)=\cos^4 x$ even fn

$\therefore \sin^7 x\cos^4 x$ is an odd fn

$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^7 x\cos^4 x\, dx=0$

</td><td>B</td></tr>
<tr><td>





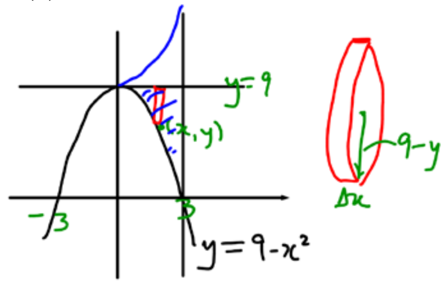
4. $\delta V=2\pi(2-x)2y\delta x$ $y^2=1-x^2$

$=4\pi(2-x)\sqrt{1-x^2}\delta x$ $y=\sqrt{1-x^2}\ \ length$

$V=\int_{-1}^{1} 4\pi(2-x)\sqrt{1-x^2}dx$

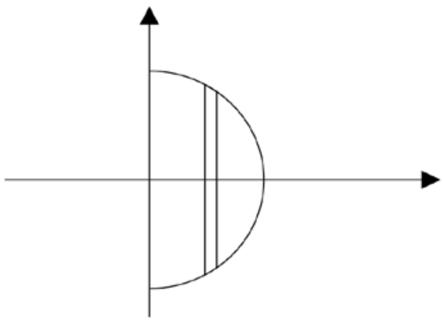
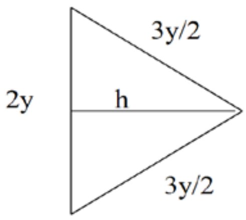
</td><td>B</td></tr>
</table>

$5(a)\quad \int_{-1}^{5}\frac{dx}{\sqrt{32+4x-x^2}}dx=\int_{-1}^{5}\frac{dx}{\sqrt{32+4-\left(x^2-4x+4\right)}}dx$ $=\int_{-1}^{5}\frac{dx}{\sqrt{36-\left(x-2\right)^2}}dx$ √ $=\left[\sin^{-1}\left(\frac{x-2}{6}\right)\right]_{-1}^{5}$ √ $=\sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(-\frac{1}{2}\right)$ $=\frac{\pi}{6}+\frac{\pi}{6}$ $=\frac{\pi}{3}$ √	**1. completing the square** **1. correct integration** **1. correct solution**
$5(b)$ $u=\tan^{-1}x \qquad v=\frac{1}{2}x^2$ $u'=\frac{1}{1+x^2} \qquad v'=x$ $\int x\tan^{-1}xdx=\frac{1}{2}x^2\tan^{-1}x-\int\frac{1}{2}x^2.\frac{1}{1+x^2}dx$ √ $=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$ $=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int 1-\frac{1}{1+x^2}dx$ √ $=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\left(x-\tan^{-1}x\right)+c$ $=\frac{1}{2}\left(x^2\tan^{-1}x-x+\tan^{-1}x\right)c$ √	**1. correct attempt at integration by parts** **2. correctly splitting x squared over 1 plus x squared** **3. correct solution**

5(c)  $V_{slice}=\pi(9-y)^2\,\Delta x$ √ $=\pi\left(9-(9-x^2)\right)^2\,\Delta x$ $=\pi x^4\Delta x$ $V=\lim_{\Delta x\to 0}\sum_{0}^{3}\pi x^4\Delta x$ $=\pi\int_0^3 x^4dx$ √ $=\pi\left[\frac{x^5}{5}\right]_0^3$ $=\pi\left(\frac{3^5}{5}-0\right)$ $=\frac{243\pi}{5}\ units^3$ √	**3. correct solution** **2. substantially correct solution** **1. correct attempt to find the volume of revolution which may include finding the volume of a slice**
5*d* $I_n=\int x^ne^xdx$ $u=x^n$ $v=e^x$ $u'=nx^{n-1}$ $v'=e^x$ $=e^xx^n-\int nx^{n-1}e^xdx$ $=e^xx^n-n\int x^{n-1}e^xdx$ $=e^xx^n-nI_{n-1}$	**1 correct show**
ii) $\int_1^2 x^2e^xdx=I_2$ $=\left[x^2e^x\right]_1^2-2\int_1^2 xe^xdx$ √ $=(4e^2-e)-2\left(\left[xe^x\right]_1^2-\int_1^2 e^xdx\right)$ $=(4e^2-e)-2(2e^2-e)+2\left[e^x\right]_1^2$ $=4e^2-e-4e^2+2e+2(e^2-e)$ $=2e^2-e$ √	**2 correct solution** **1. correct application of the recurrence rule above**

$e)\ \int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin x+3\cos x}$ $t=\tan\left(\frac{x}{2}\right)$ $=\int_0^1 \frac{1}{5+4\left(\frac{2t}{1+t^2}\right)+3\left(\frac{1-t^2}{1+t^2}\right)}.\left(\frac{2}{1+t^2}\right)dt$ √ $x=2\tan^{-1}t$ $=\int_0^1 \frac{2}{5(1+t^2)+4(2t)+3(1-t^2)}\ dt$ $dx=\left(\frac{2}{1+t^2}\right)dt$ $=\int_0^1 \frac{2}{5+5t^2+8t+3-3t^2}\ dt$ √ $x=0,\ t=0$ $=\int_0^1 \frac{2}{2t^2+8t+8}\ dt$ $x=\frac{\pi}{2},\quad t=1$ $=\int_0^1 \frac{2}{2(t+2)^2}\ dt$ $=\left[\frac{-1}{t+2}\right]_0^1$ $=\frac{-1}{3}+\frac{1}{2}$ $=\frac{1}{6}$ √	**3. correct solution** **2. Substantially correct** **1. Correct attempt to use the t substitution**
Question 6	
$a)i)\quad \frac{10x}{(x^2+1)(3x+1)}=\frac{x+A}{(x^2+1)}+\frac{B}{3x+1}$ $10x=(x+A)(3x+1)+B(x^2+1)$ coeff of x^2 : $0=3+B$ $\therefore B=-3$ constant term: $0=A+B$ $\therefore A=3$	**2 both correct** **1 either A or B correct**

ii) $\int_0^2 \frac{10x}{(x^2+1)(3x+1)}dx = \int_0^2 \frac{x+3}{(x^2+1)} - \frac{3}{3x+1}dx$ $= \int_0^2 \frac{x}{(x^2+1)} + \frac{3}{(x^2+1)} - \frac{3}{3x+1}dx$ $= \left[\frac{1}{2}\ln\left\|x^2+1\right\| + 3\tan^{-1}x - \ln\left\|3x+1\right\|\right]_0^2$ $= \frac{1}{2}\ln 5 + 3\tan^{-1}2 - \ln 7 - \frac{1}{2}\ln 1 + 3\tan^{-1}0 - \ln 1$ $= 2.18025....$ $= 2.18 \text{ m}^2$	**2. correct solution** **1. for splitting the fraction in parts that can be integrated**
b)i) To show $\frac{x}{x^2+4} \le \frac{1}{4}$ we show $\frac{x}{x^2+4} - \frac{1}{4} \le 0$ $\frac{x}{x^2+4} - \frac{1}{4} = \frac{4x - 1(x^2+4)}{4(x^2+4)}$ $= \frac{4x - x^2 - 4}{4(x^2+4)}$ $= \frac{-(x-2)^2}{4(x^2+4)}$ ≤ 0 as $(x-2)^2 \ge 0,\quad x^2 \ge 0 \;\therefore x^2+4 \ge 4 > 0$	**2 correct solution including justification for inequality** **1. correct attempt to show without correct justification of inequatlity**
ii)	**2 correct solutions** **1**

$\frac{x}{x^2+4} \le \frac{1}{4}$ $\int_0^\alpha \frac{x}{x^2+4}dx \le \int_0^\alpha \frac{1}{4}dx$ $\left[\frac{1}{2}\ln\left\|x^2+4\right\|\right]_0^\alpha \le \left[\frac{x}{4}\right]_0^\alpha$ $\frac{1}{2}\left(\ln\left\|\alpha^2+4\right\| - \ln\left\|4\right\|\right) \le \frac{\alpha}{4} - 0$ $\ln\left(\alpha^2+4\right) - \ln 4 \le \frac{\alpha}{2} \qquad \alpha^2+4 > \alpha^2 \ge 0$ $\ln\left(\frac{\alpha^2+4}{4}\right) \le \frac{\alpha}{2}$ $\therefore \frac{\alpha^2+4}{4} \le e^{\frac{\alpha}{2}}$ $\frac{\alpha^2}{4} + \frac{4}{4} \le e^{\frac{\alpha}{2}}$ $\therefore e^{\frac{\alpha}{2}} \ge \frac{1}{4}\alpha^2 + 1$	
C) i)  By Pythagoras	**3** **2** **1**

$h^2 + y^2 = \left(\frac{3y}{2}\right)^2$ $h^2 = \frac{9y^2}{4} - y^2$ $= \frac{5y^2}{4}$ $h = \frac{\sqrt{5}y}{2}$ length >0 when $x = a$ $a^2 + y^2 = 1$ $y^2 = 1 - a^2$ Area of triangle= $\frac{1}{2}.2y.\frac{\sqrt{5}y}{2}$ $= \frac{\sqrt{5}y^2}{2}$ $= \frac{\sqrt{5}}{2}\left(1 - a^2\right)$	
ii) As $x = a$ where $0 < a < 1$ Volume slice = $\frac{\sqrt{5}}{2}\left(1 - x^2\right)\delta x$ Volume of solid = $\lim_{\delta x \to 0} \sum_{x=0}^{1} \frac{\sqrt{5}}{2}\left(1 - x^2\right)\delta x$ $= \frac{\sqrt{5}}{2}\int_0^1 \left(1 - x^2\right) dx$ $= \frac{\sqrt{5}}{2}\left[x - \frac{x^3}{3}\right]_0^1$ $= \frac{\sqrt{5}}{2}\left(1 - \frac{1}{3} - 0\right)$ $= \frac{\sqrt{5}}{3}$ $units^3$	**2 Correct** **1 correct attempt**